## Aufgabe 1:

a)

$$\vec{E} = E_0 \vec{e}_x$$

Symmetrien: Translationssymmetrie entlang der $z$-Achse:

$$\vec{E} = \vec{E}(r, \varphi)$$

Das Potential ist also auch unabhängig von $z \Rightarrow E_z = -\partial_z \Phi = 0$ Spiegelsymmetrie bzgl. der $x$-Achse $\vec{E}(r, \varphi) = \vec{E}(r, -\varphi)$

b)

$$\Delta\Phi(r, \varphi) = \frac{1}{r}\partial_r r \partial_r \Phi + \frac{1}{r^2}\partial_\varphi^2 \Phi = 0$$

NR:

$$(\partial_x + \partial_y)\Phi(x, y) = 0$$

$$\Phi(x, y) = A(x)B(y) \quad , \quad \partial_x \partial_y f = \partial_y \partial_x f$$

$$0 = \partial_x A(x)B(y) + \partial_y A(x)B(y) = B(y)\partial_x A(x) + A(x)\partial_y B(y)$$

$$\Rightarrow 0 = \frac{1}{A(x)}\partial_x A(x) + \frac{1}{B(y)}\partial_y B(y)$$

$$\partial_x A(x) = A(x)c \Rightarrow A(x) = A_0 e^{cx} \quad , \quad \partial_y B(y) = -B(y)c \Rightarrow B(y) = B_0 e^{-cx}$$

$$\Phi(x, y) = \Phi_0 e^{c(x-y)}$$

$$\Phi(r, \varphi) = U(r)\chi(\varphi)$$

Mit Laplace in Zylkos.

$$\Rightarrow \frac{r}{U(r)}\partial_r r \partial_r U(r) = -\frac{1}{\chi(\varphi)}\partial_\varphi^2 \chi(\varphi)$$

$$\Rightarrow \frac{r}{U(r)}\partial_r r \partial_r U(r) = \mu^2; \partial_\varphi^2 \chi(\varphi) = -\mu^2 \chi(\varphi)$$

$$\Rightarrow \chi(\varphi) = A_+ e^{i\mu\varphi} + A_- e^{-i\mu\varphi}$$

$$\chi(\varphi + 2\pi) = \chi(\varphi) \quad , \quad e^{i\mu(\varphi+2\pi)} = e^{i\mu\varphi}$$

$$\Rightarrow e^{i\mu 2\pi} = 1 \Rightarrow \mu \in \mathbb{Z}$$

$$\Rightarrow \chi(\varphi) = A_{s,n} \sin n\varphi + A_{c,n} \cos n\varphi \ , \ n \in \mathbb{Z}$$

$$r\partial_r r \partial_r U(r) = \mu^2 U(r)$$

Ansatz mit der Annahme $U$ analytisch

$$U(r) = \sum_{n=-\infty}^{\infty} c_n r^n$$

$$\partial_r U(r) = U' = \sum_{n=-\infty}^{\infty} n c_n r^{n-1}$$

$$rU' = \sum_{n=-\infty}^{\infty} n c_n r^n \quad , \quad r\partial_r U' = \sum_{n=-\infty}^{\infty} c_n n^2 r^n$$

$$\sum_{n=0}^{\infty}(n^2c_n - \mu^2 c_n) = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} c_n(n^2-\mu^2)r^n = 0 \Rightarrow \begin{cases} n = \pm\mu = k \in \mathbb{Z}\ ,\ c_k \text{ beliebig}, c_n = 0\ \forall n \neq k \\ \qquad (c_n = 0\ \forall n \in \mathbb{Z}) \end{cases}$$

$$\Rightarrow U(r) = c_k r^k + c_{-k} r^{-k}$$

$$\chi(\varphi) = A_n \sin n\varphi + B_n \cos n\varphi$$

$$U(r) = C_n r^n + C_{-n} r^{-n}$$

$$\partial_r r \partial_r U(r) = 0(n = 0 : U(r) = \ln\frac{r}{c_0} \Rightarrow \Delta\Phi \propto \delta(r)$$

$$\Phi(r,\varphi) = A_0 + \sum_{n=1}^{\infty}(A_{c,n}\cos n\varphi + A_{s,n}\sin n\varphi)\left(B_n r^n + B_{-n} r^{-n}\right)$$

Randbedingungen

1) $\Phi(r,\varphi) = \Phi(r,-\varphi)$
2) $\Phi(R,\varphi) = 0\ \forall\varphi \in [0,2\pi]$ (Aus Bequemlichkeit)
3) $\lim_{r\to\infty}\vec{\nabla}\Phi(r,\varphi) = -\vec{E}_0$

$$\Phi(R,\varphi) = A_0 + \sum_{n=1}^{\infty} A_{c,n}\cos n\varphi(B_n R^n + B_{-n}R^{-n}) \stackrel{!}{=} 0$$

- $A_{c,n} = 0 (n \in \mathbb{N}) \Rightarrow$ triviale Lösung ↯
- $A_0 = 0;\ B_n R^n + B_{-n}R^{-n} = 0 \Rightarrow B_{-n} = -B_n R^{2n}$

$$\Rightarrow \Phi(r,\varphi) = \sum_{n=1}^{\infty} A_{c,n}\cos n\varphi R^n B_n\left(\left(\frac{r}{R}\right)^n - \left(\frac{R}{r}\right)^n\right)$$

$$A_n := A_{c,n}R^n B^n$$

- $\vec{\nabla} = (\partial_r, \frac{1}{r}\partial_\varphi)\ ,\ \vec{E} = E_0\vec{e}_x$

$$\vec{e}_x = \cos\varphi\vec{e}_r - \sin\varphi\vec{e}_\varphi$$

$$\lim_{r\to\infty}((\partial_r\Phi)\vec{e}_r + (\frac{1}{r}\partial_\varphi\Phi)\vec{e}_\varphi) = -E_0\cos\varphi\vec{e}_r + E_0\sin\varphi\vec{e}_\varphi$$

Feld ist im Unendlichen konstant $\Rightarrow$ Potential muss dort linearen Zusammenhang mit $r$ haben

$$= A_1\left(\frac{r}{R} - \frac{R}{r}\right)\cos\varphi = \frac{A_1}{R}\left(r - \frac{R^2}{r}\right)\cos\varphi \Rightarrow \frac{A_1}{R} = -E_0$$

- Das elektrische Feld im Inneren des Leiters ist 0.

c)

$$E_n|_{r=R} = -\left.\frac{\partial\Phi}{\partial\vec{b}}\right|_{r=R} = -(\vec{n}\cdot\vec{\nabla})\Phi = -4\pi\sigma_{inf}$$

$$\vec{n} = \vec{e}_r \quad ; \quad \vec{n}\cdot\vec{\nabla} = \frac{\partial}{\partial r} \quad ; \quad \Phi = -E_0\cos\varphi(r - \frac{R^2}{r}) \text{ mit } r \geq R$$

$$\sigma_{inf} = \frac{1}{2\pi}E_0\cos\varphi$$

$$Q_{inf} \sim \int_0^{2\pi} \mathrm{d}\varphi \sigma_{inf} = 0$$

$$Q = Q_+ + Q_- \Rightarrow Q_+ = -Q_-$$

$$Q \geq 0 : \varphi \in \{-\frac{\pi}{2}, \frac{\pi}{2}\}$$

$$Q \leq 0 : \varphi \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$$

$$\frac{Q_+}{z} = \frac{-Q_-}{z} = R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\varphi \sigma_{inf} = \frac{R\,E_0}{\pi}$$

$$\rho_+(\vec{x}) = \frac{1}{2\pi} E_0 \cos\varphi \delta(r - R)$$

$$\vec{R}_+ Q_+ = \int \mathrm{d}^3 x \vec{x} \rho_+(\vec{x}) = \frac{R^2 E_0}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\varphi x r \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \cos\varphi = \frac{R^2 E_0}{4} \vec{e}_X$$

$$\vec{R}_+ = \frac{\pi}{4} R$$

$$\vec{R}_- = -\vec{R}_+$$

$$\frac{\vec{d}}{z} == \vec{R}_+ \frac{Q_+}{z} + \vec{R}_- \frac{Q_-}{z} = \frac{R^2 E_0}{2} \vec{e}_x$$